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Two-qubit entanglement in an antiferromagnetic environment

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Abstract. Using the spin wave approximation, we study the entanglement of a two-spin system in an antiferromagnetic environment. It is shown analytically that the concurrence of the two-spin system displays a Gaussian decay and depends on the quantum correlations and the structure of the environment. The relation between the concurrence and the environmental temperature is given explicitly.

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Entanglement is one of the distinguishing properties of quantum mechanics and shows potential applications in quantum communication and information processing [1–5]. Because of this reason, the entanglement between two qubits deserves to be analyzed in all respects, especially the influences of the environment which cause the decoherence of the qubits. The dynamic evolution of a small quantum system interacting with large open environment has attracted much attention in recent years. As one aspect of such problem, some authors [6–8] considered a single spin or several spins interacting with a spin environment. Although useful results have been obtained, they made assumptions about the distribution of the spin states of the environment or neglected the environmental dynamics. This is not the real case. In fact, the environment has structure and certain quantum correlations may exist among spins in the environment. Recently, Lucamarini et al. [9] considered the influence of the environment on the concurrence [10] of a two-spin system. The environment was described by Ising model and the mean field approximation was used. The authors calculated the concurrence which reflects the entanglement degree of the two-spin system. It is not a trivial work to consider other kinds of the environment such as an antiferromagnetic environment. The concurrence of two spins embedded in an antiferromagnetic environment has been calculated [11,12]. However, for simplicity, the environment was assumed to be spin chains and the time evolution of the concurrence was not given. In the present paper, we will continue the study and use the spin wave approximation to deal with an antiferromagnetic environment of any dimensions. In comparison with the mean field approximation, the spin wave approximation has more advantages [13]. We show analytically that the concurrence displays a Gaussian decay and depends on the quantum correlations and the structure of the environment. The relation between the concurrence and the environmental temperature is given explicitly.

We consider a two-spin system coupling with an antiferromagnetic environment. The system and the environment consist of spin-$\frac{1}{2}$ atoms. H_S and H_B are the Hamiltonians of the system and the environment respectively, and H_{SB} is the coupling term [9,13]:

$$H = H_S + H_{SB} + H_B, \tag{1}$$

where

$$H_S = -\mu_0 \left(S_{01}^z + S_{02}^z\right) - \xi_0 S_{01}^z S_{02}^z, \tag{2}$$

$$H_{SB} = -\frac{J_0}{\sqrt{N}} \left(S_{01}^z + S_{02}^z\right) \sum_i \left(S_{a,i}^z + S_{b,i}^z\right), \tag{3}$$

$$H_B = J \sum_{i,\vec{\delta}} \mathbf{S}_{a,i} \cdot \mathbf{S}_{b,i+\vec{\delta}} + J \sum_{j,\vec{\delta}} \mathbf{S}_{b,j} \cdot \mathbf{S}_{a,j+\vec{\delta}}, \tag{4}$$

where μ_0 is the coupling constant with an external magnetic field in the z direction. ξ_0 represents the coupling constant between the two spins. J_0 is the exchange coupling constant between the system and its environment, whereas J is that in the environment and is positive for antiferromagnets. The effects of next-nearest-neighbor interactions are neglected, although they may be important in real antiferromagnets. We assume that the spin structure of the environment may be divided into two interpenetrating sublattices a and b with the property that all nearest neighbors of an atom on a lie on b, and vice versa [13]. $\mathbf{S}_{a,i}$ ($\mathbf{S}_{b,j}$) represents the spin operator of the ith (jth) atom on sublattice a (b). Each sublattice contains N atoms. The indices i and j label the N atoms, whereas the vectors $\vec{\delta}$ connect atom i or j with its nearest neighbors.

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We use the spin wave approximation [13] to map spin operators of the environment onto boson operators. After the Holstein-Primakoff transformation:

$$S_{a,i}^{+} = \sqrt{1-a_i^{+}a_i}a_i, \quad S_{a,i}^{-} = a_i^{+}\sqrt{1-a_i^{+}a_i},$$
$$S_{a,i}^{z} = \frac{1}{2} - a_i^{+}a_i, \tag{5}$$

$$S_{b,j}^{+} = b_j^{+}\sqrt{1-b_j^{+}b_j}, \quad S_{b,j}^{-} = \sqrt{1-b_j^{+}b_j}b_j,$$
$$S_{b,j}^{z} = b_j^{+}b_j - \frac{1}{2}, \tag{6}$$

with low-lying states of the environment, the Hamiltonians H_{SB} and H_B are written as

$$H_{SB} = -\frac{J_0}{\sqrt{N}}\left(S_{01}^z + S_{02}^z\right)\sum_i \left(b_i^{+}b_i - a_i^{+}a_i\right), \tag{7}$$

$$H_B = -\frac{1}{2}NMJ + MJ\sum_i \left(a_i^{+}a_i + b_i^{+}b_i\right) + J\sum_{i,\vec{\delta}}\left(a_i b_{i+\vec{\delta}} + a_i^{+}b_{i+\vec{\delta}}^{+}\right), \tag{8}$$

where M is the number of the nearest neighbors of an atom. By transforming to the momentum space, we have

$$H_{SB} = -\frac{J_0}{\sqrt{N}}\left(S_{01}^z + S_{02}^z\right)\sum_{\mathbf{k}} \left(b_{\mathbf{k}}^{+}b_{\mathbf{k}} - a_{\mathbf{k}}^{+}a_{\mathbf{k}}\right), \tag{9}$$

$$H_B = -\frac{1}{2}NMJ + MJ\sum_{\mathbf{k}} \left(a_{\mathbf{k}}^{+}a_{\mathbf{k}} + b_{\mathbf{k}}^{+}b_{\mathbf{k}}\right) + MJ\sum_{\mathbf{k}}\gamma_{\mathbf{k}}\left(a_{\mathbf{k}}^{+}b_{\mathbf{k}}^{+} + a_{\mathbf{k}}b_{\mathbf{k}}\right), \tag{10}$$

where $\gamma_{\mathbf{k}} = M^{-1}\sum_{\vec{\delta}} e^{i\mathbf{k}\cdot\vec{\delta}}$. Then by using the Bogoliubov transformation, the Hamiltonians H_{SB} and H_B can be diagonalized ($\hbar = 1$):

$$H_{SB} = -\frac{J_0}{\sqrt{N}}(S_{01}^z + S_{02}^z)\sum_{\mathbf{k}}(\beta_{\mathbf{k}}^{+}\beta_{\mathbf{k}} - \alpha_{\mathbf{k}}^{+}\alpha_{\mathbf{k}}), \tag{11}$$

$$H_B = -\frac{3}{2}NMJ + \sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\alpha_{\mathbf{k}}^{+}\alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^{+}\beta_{\mathbf{k}} + 1\right), \tag{12}$$

where $\alpha_{\mathbf{k}}^{+}$ ($\alpha_{\mathbf{k}}$) and $\beta_{\mathbf{k}}^{+}$ ($\beta_{\mathbf{k}}$) are the creation (annihilation) operators of the two different magnons with wavevector $\mathbf{k}$ and frequency $\omega_{\mathbf{k}}$ respectively. For cubic crystal system,

$$\omega_{\mathbf{k}} = (2M)^{1/2}Jkl, \tag{13}$$

where l is the side length of cubic primitive cell of sublattice. This means that the magnon energy increases linearly with k for antiferromagnets.

We assume the initial density matrix of the system-environment to be factorized, i.e., $\rho(0) = |\psi\rangle\langle\psi| \otimes \rho_B$. The initial state of the two-spin system is

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle, \tag{14}$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1, \tag{15}$$

where $|0\rangle$ and $|1\rangle$ are the lower and upper eigenstates of S_0^z respectively. The density matrix of the environment satisfies a thermal distribution, that is $\rho_B = e^{-H_B/T}/Z$, where Z is the partition function and the Boltzmann constant has been set to one. After tracing over the environmental degrees of freedom, we obtain the reduced density matrix of the system:

$$\rho_s(t) = \mathrm{tr}_B\left\{e^{-iHt}\left[|\psi\rangle\langle\psi| \otimes \rho_B\right]e^{iHt}\right\}. \tag{16}$$

The matrix $\rho_s(t)$ is a 4×4 matrix in the standard basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. As an example, we just illustrate the calculation of one element of the matrix [14]:

$$[\rho_s(t)]_{12} = \langle 00|\rho_s(t)|01\rangle = \frac{\alpha\beta^*}{Z}\mathrm{tr}_B\left[e^{-i(\mu_0-\xi_0/2)t - i\frac{J_0}{\sqrt{N}}\sum_{\mathbf{k}}(\beta_{\mathbf{k}}^{+}\beta_{\mathbf{k}} - \alpha_{\mathbf{k}}^{+}\alpha_{\mathbf{k}})t}e^{-H_B/T}\right], \tag{17}$$

where

$$Z = \mathrm{tr}_B e^{-H_B/T} = \mathrm{tr}_B e^{3NMJ/2T - \sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\alpha_{\mathbf{k}}^{+}\alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^{+}\beta_{\mathbf{k}} + 1\right)/T} = e^{3NMJ/2T}\prod_{\mathbf{k}} e^{-\omega_{\mathbf{k}}/T}\left(\prod_{\mathbf{k}}\frac{1}{1-e^{-\omega_{\mathbf{k}}/T}}\right)^2, \tag{18}$$

$$\mathrm{tr}_B\left[e^{-i(\mu_0-\xi_0/2)t - i\frac{J_0}{\sqrt{N}}\sum_{\mathbf{k}}(\beta_{\mathbf{k}}^{+}\beta_{\mathbf{k}} - \alpha_{\mathbf{k}}^{+}\alpha_{\mathbf{k}})t}e^{-H_B/T}\right] = e^{-i(\mu_0-\xi_0/2)t}e^{3NMJ/2T}\prod_{\mathbf{k}} e^{-\omega_{\mathbf{k}}/T}\prod_{\mathbf{k}}\frac{1}{1-e^{i\frac{J_0}{\sqrt{N}}t}e^{-\omega_{\mathbf{k}}/T}} \times \prod_{\mathbf{k}}\frac{1}{1-e^{-i\frac{J_0}{\sqrt{N}}t}e^{-\omega_{\mathbf{k}}/T}}. \tag{19}$$

Therefore, we get

$$[\rho_s(t)]_{12} = \alpha\beta^* e^{-i(\mu_0-\xi_0/2)t}\frac{y_1 y_2}{y^2}, \tag{20}$$

where

$$y = \prod_{\mathbf{k}}\frac{1}{1-e^{-\omega_{\mathbf{k}}/T}}, \tag{21}$$

$$y_1 = \prod_{\mathbf{k}}\frac{1}{1-e^{i\frac{J_0}{\sqrt{N}}t}e^{-\omega_{\mathbf{k}}/T}}, \tag{22}$$

$$y_2 = \prod_{\mathbf{k}}\frac{1}{1-e^{-i\frac{J_0}{\sqrt{N}}t}e^{-\omega_{\mathbf{k}}/T}}. \tag{23}$$

From equation (21), we arrive at

$$\ln y = -\sum_{\mathbf{k}}\ln\left(1-e^{-\omega_{\mathbf{k}}/T}\right) = -\frac{V}{8\pi^3}\int \ln\left(1-e^{-\omega_{\mathbf{k}}/T}\right)4\pi k^2 dk, \tag{24}$$

where V is the volume of the environment. At low temperature such that $(\omega_{\mathbf{k}})_{\max} \gg T$, with $x = (2M)^{1/2}Jkl/T$,

$$\ln y = -\frac{NT^3}{4\sqrt{2}\pi^2 M^{3/2}J^3}\int_0^\infty \ln\left(1-e^{-x}\right)x^2 dx. \tag{25}$$

$$\rho_s(t) = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i(\mu_0-\xi_0/2)t}A & \alpha\gamma^* e^{-i(\mu_0-\xi_0/2)t}A & \alpha\delta^* e^{-i2\mu_0 t}A^4 \\ \alpha^*\beta e^{i(\mu_0-\xi_0/2)t}A & |\beta|^2 & \beta\gamma^* & \beta\delta^* e^{-i(\mu_0+\xi_0/2)t}A \\ \alpha^*\gamma e^{i(\mu_0-\xi_0/2)t}A & \beta^*\gamma & |\gamma|^2 & \gamma\delta^* e^{-i(\mu_0+\xi_0/2)t}A \\ \alpha^*\delta e^{i2\mu_0 t}A^4 & \beta^*\delta e^{i(\mu_0+\xi_0/2)t}A & \gamma^*\delta e^{i(\mu_0+\xi_0/2)t}A & |\delta|^2 \end{pmatrix} \quad (32)$$

In the same way, we have

$$\ln y_1 = -\frac{NT^3}{4\sqrt{2}\pi^2 M^{3/2}J^3}\int_0^\infty \ln\left(1-e^{i\theta}e^{-x}\right)x^2dx, \quad (26)$$

$$\ln y_2 = -\frac{NT^3}{4\sqrt{2}\pi^2 M^{3/2}J^3}\int_0^\infty \ln\left(1-e^{-i\theta}e^{-x}\right)x^2dx, \quad (27)$$

where

$$\theta = \frac{J_0}{\sqrt{N}}t. \quad (28)$$

Let

$$f(\theta) = \int_0^\infty \ln\left(1-e^{i\theta}e^{-x}\right)x^2dx + \int_0^\infty \ln\left(1-e^{-i\theta}e^{-x}\right)x^2dx - 2\int_0^\infty \ln\left(1-e^{-x}\right)x^2dx, \quad (29)$$

it is obvious $f(\theta)\to 0$ as $\theta\to 0$.

To find the limit relation between θ and $f(\theta)$, we calculate

$$\begin{aligned}\eta &= \lim_{\theta\to 0}\frac{f(\theta)}{\theta^2} \\ &= \lim_{\theta\to 0}\frac{f'(\theta)}{2\theta} \\ &= \lim_{\theta\to 0}\frac{\int_0^\infty \frac{-ie^{i\theta}e^{-x}x^2}{1-e^{i\theta}e^{-x}}dx + \int_0^\infty \frac{ie^{-i\theta}e^{-x}x^2}{1-e^{-i\theta}e^{-x}}dx}{2\theta} \\ &= \int_0^\infty \frac{x^2e^{-x}}{(1-e^{-x})^2}dx = \frac{\pi^2}{3}. \end{aligned} \quad (30)$$

If N is large enough, $\theta\to 0$ and $f(\theta)=\eta\theta^2$. Finally, we have

$$\begin{aligned}[\rho_s(t)]_{12} &= \alpha\beta^* e^{-i(\mu_0-\xi_0/2)t}e^{-\frac{J_0^2T^3}{12\sqrt{2}M^{3/2}J^3}t^2} \\ &= \alpha\beta^* e^{-i(\mu_0-\xi_0/2)t}e^{-\frac{t^2}{\tau_0^2}}, \end{aligned} \quad (31)$$

where the decoherence time $\tau_0 = 2\sqrt{3\sqrt{2}M^{3/2}J^3/J_0^2T^3}$. In the same way, we get the other elements and write the reduced density matrix as

see equation (32) above,

where $A = e^{-t^2/\tau_0^2}$.

We use the concurrence to measure the entanglement between two spins. It ranges from 0 for separable states to 1 for maximally entangled states. The concurrence is defined as [10]

$$C_{12} = \max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4, 0\}, \quad (33)$$

where the quantities $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4$ are the square roots of the eigenvalues of the operator

$$R_{12} = \rho_s\left(\sigma^y\otimes\sigma^y\right)\rho_s^*\left(\sigma^y\otimes\sigma^y\right). \quad (34)$$

Assuming the initial state of the two-spin system as $|\psi\rangle = \alpha|00\rangle + \delta|11\rangle$, we obtain the concurrence

$$C_{12} = 2|\alpha||\delta|e^{-\frac{J_0^2T^3}{3\sqrt{2}M^{3/2}J^3}t^2}. \quad (35)$$

From equation (35) we can clearly see that the concurrence displays a Gaussian decay. The factor t^2 in the exponent denotes the intrinsically reversible nature of the process, in contrast to irreversibility introduced by Markovian approach. With the increase of the exchange coupling constant J, the decay rate decreases. It means that the strong quantum correlations in the environment limit the decay of the concurrence. This is quite easy to understand. The strong quantum correlations in the environment allow energy exchange without using the system as an intermediary. Without the correlations, the energy must flow through the system, leaving the entanglement badly spoiled. The structure of the environment also influences the concurrence of the system, since large coordination number can suppress the decay rate of the concurrence. It is obvious that the concurrence is sensitive to the environmental temperatures. At absolute zero temperature, the entanglement is independent on time, so the system does not perceive the presence of the environment. Certainly, here we do not consider the effect of quantum fluctuations. It should be emphasized that the main differences of our work from those in reference [9] lie in that the environment is described by Heisenberg model rather than Ising model. We use the spin wave approximation rather than the mean field approximation. The relation between the concurrence and the environmental temperature is obtained explicitly. Consequently, our results are more reasonable. Finally, for visualization, we plot the concurrence as a function of time according to equation (35). Figure 1 shows the time evolution of the concurrence for three different temperatures. Initially, the two-spin system is in a maximally entangled state. In terms of the influence of the environment, the entanglement between two spins is spoiled. At high temperature, the entanglement disappears quickly as shown in the figure.

In conclusion, we have studied the entanglement of two-spin system embedded in an antiferromagnetic environment with the help of the spin wave approximation.

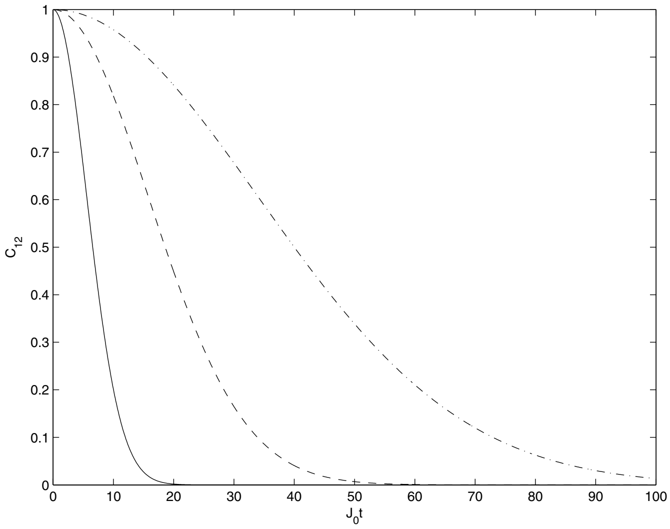

Fig. 1. Time evolution of the concurrence for three different temperatures. $\alpha = \delta = \sqrt{2}/2$, $M = 6$, $T = J$ (solid curve), $T = 0.5J$ (dashed curve), and $T = 0.3J$ (dot-dashed curve).

The analytical results show that the concurrence displays a Gaussian decay. With the increase of temperature, the decay rate of the concurrence increases. Furthermore, the strong quantum correlations and the large coordination number in the environment limit the decay rate of the concurrence. We are sure that the present theoretical study will shed light on the applications of spin system in the region of quantum communication and computation.

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